# TCBB paper - Supplementary materials

## 1 Supplemental material 1

**Theorem 1.** *Let $\mathcal{B} = \langle V, F, d\rangle$ be a BSBN and $\mathcal{D}$ be its encoded DGARBN by Definition 5. For any pair of states $s$ and $s'$, we have $s'$ is reachable from $s$ in $\mathcal{B}$ iff $[[s']]^{\mathcal{D}}$ is reachable from $[[s]]^{\mathcal{D}}$ in $\mathcal{D}$.*

*Proof.* Assume that $s'$ is the next state of $s$ in $\mathcal{B}$, i.e., $s \xrightarrow{\mathcal{B}} s'$. Then, $s \xrightarrow{B_1} s^1 ... s^{m-1} \xrightarrow{B_m} s'$ where $m = nb(d)$ and $B_i$ is the $i$-th block of $d$ and $s \xrightarrow{B_1} s^1$ denotes that $s^1$ is the state obtained by updating all nodes of $B_1$ in parallel with the current state is $s$. In $[[s]]^{\mathcal{D}}$, the nodes of $B_1$ will be updated since $t_{scaled} = 0, t_{scaled} \% p_i = 0 = q_i, x_i \in B_1$. Then, $[[s]]^{\mathcal{D}} \xrightarrow{\mathcal{D}} (s^1, 1)$ where $(s^1, 1)$ is an extended state of $\mathcal{D}$ with $t_{scaled} = 1$. Similarly, we have $[[s]]^{\mathcal{D}} \xrightarrow{\mathcal{D}} (s^1, 1)...(s^{m-1}, m-1) \xrightarrow{\mathcal{D}} [[s']]^{\mathcal{D}}$. That means $[[s']]^{\mathcal{D}}$ is reachable from $[[s]]^{\mathcal{D}}$ in $\mathcal{D}$.

Assume that $[[s]]^{\mathcal{D}} \xrightarrow{\mathcal{D}} (s^1, 1)...(s^{m-1}, m-1) \xrightarrow{\mathcal{D}} [[s']]^{\mathcal{D}}$. Let $V_1$ be the set of nodes whose updates make $\mathcal{D}$ transits from $[[s]]^{\mathcal{D}}$ to $(s^1, 1)$. Then, $t_{scaled} \% p_i = 0 = q_i, i \in V_1$. Thus, $V_1 = B_1$, implying that $s \xrightarrow{B_1} s^1$ in $\mathcal{B}$. Similarly, we have $s \xrightarrow{B_1} s^1 ... s^{m-1} \xrightarrow{B_m} s'$. That means $s'$ is reachable from $s$ in $\mathcal{B}$.

Now, we can conclude the proof. □

**Theorem 2.** *Let $\mathcal{B} = \langle V, F, d\rangle$ be a BSBN and $\mathcal{D}$ be its encoded DGARBN by Definition 5. Let $A^{\mathcal{B}}$ and $A^{\mathcal{D}}$ be the sets of attractors of $\mathcal{B}$ and $\mathcal{D}$, respectively. Then, $A^{\mathcal{B}}$ one-to-one corresponds to $A^{\mathcal{D}}$. Moreover, given an attractor att of $\mathcal{B}$, its corresponding attractor of $\mathcal{D}$ is $att'$ satisfying $att = [[att']]^{\mathcal{B}}$ where $[[ES]]^{\mathcal{B}} = \{s | (s, 0) \in ES\}$, $ES$ is a set of extended states.*

*Proof.* Assume that $att$ is an attractor of $\mathcal{B}$. Let $FR^{\mathcal{B}}(S)$ denote the set of states reachable from the set $S$ of states in $\mathcal{B}$. Let $s$ be a state in $att$. Then, $FR^{\mathcal{B}}(\{s\}) = att$. Let $FR^{\mathcal{D}}(ES)$ denote the set of extended states reachable from the set $ES$ of extended states in $\mathcal{D}$. Start from $[[s]]^{\mathcal{D}}$, following the evolution of $\mathcal{D}$, we will come back to $[[s]]^{\mathcal{D}}$ by Theorem 1. Since $\mathcal{D}$ has only fixed points or limit cycles, $FR^{\mathcal{D}}(\{[[s]]^{\mathcal{D}}\})$ is an attractor of $\mathcal{D}$. Moreover, $[[FR^{\mathcal{D}}(\{[[s]]^{\mathcal{D}}\})]]^{\mathcal{B}} = FR^{\mathcal{B}}(\{s\})$ by Theorem 1. Thus, $[[FR^{\mathcal{D}}(\{[[s]]^{\mathcal{D}}\})]]^{\mathcal{B}} = att$. (i)

Given $att_1, att_2 \in A^{\mathcal{B}}, att_1 \cap att_2 = \emptyset$. Consider an arbitrary pair of states $s^1 \in att_1$ and $s^2 \in att_2$. If $FR^{\mathcal{D}}(\{[[s^1]]^{\mathcal{D}}\}) \cap FR^{\mathcal{D}}(\{[[s^2]]^{\mathcal{D}}\}) \neq \emptyset$, then they are the same attractor since $\mathcal{D}$ has only fixed points or limit cycles. This implies that $att_1 = att_2$ contradicting to $att_1 \cap att_2 = \emptyset$. Thus, $FR^{\mathcal{D}}(\{[[s^1]]^{\mathcal{D}}\}) \cap FR^{\mathcal{D}}(\{[[s^2]]^{\mathcal{D}}\}) = \emptyset$. (ii)

Assume that $att'$ is an attractor of $\mathcal{D}$. Let $att$ be a set of states in $\mathcal{B}$ such that $att = [[att']]^{\mathcal{B}}$. By Theorem 1, each state of $att$ is reachable from any other state of $att$. Suppose that there exists a state $s \notin att$ such that it is reachable from $att$. Then $[[s]]^{\mathcal{D}}$ is not in $att'$ but is reachable from $att'$ by Theorem 1. This is a contradiction. Thus, $att$ is an attractor of $\mathcal{B}$. (iii)

Given $att'_1, att'_2 \in A^{\mathcal{D}}, att'_1 \cap att'_2 = \emptyset$. Oviously, $[[att'_1]]^{\mathcal{B}} \cap [[att'_2]]^{\mathcal{B}} = \emptyset$. (iv)

There is an injection from $A^{\mathcal{B}}$ to $A^{\mathcal{D}}$ by (i) and (ii). There is also an injection from $A^{\mathcal{D}}$ to $A^{\mathcal{B}}$ by (iii) and (iv). Thus, $A^{\mathcal{B}}$ one-to-one corresponds to $A^{\mathcal{D}}$. In addition, $att = [[att']]^{\mathcal{B}}$ with $att$ is an attractor of $\mathcal{B}$ and $att'$ is its corresponding attractor of $\mathcal{D}$. We also obtain that $|att'| = nb(d) \times |att|$ since $\mathcal{D}$ has only fixed points or limit cycles. □

**Lemma 1.** *Let $\mathcal{D}$ be a DGARBN and $\mathcal{G}$ be its GARBN counterpart. Let es be an extended state of $\mathcal{D}$ and $FR^{\mathcal{D}}(\{es\})$ be the set of extended states reachable from es in $\mathcal{D}$. Then, $[[FR^{\mathcal{D}}(\{es\})]]^{I} \subseteq FR^{\mathcal{G}}(\{[[es]]^{I}\})$.*

*Proof.* Let $G(\mathcal{D})$ and $G(\mathcal{G})$ be the ESTG and STG of $\mathcal{D}$ and $\mathcal{G}$, respectively. If $(es, es')$ is an arc in $G(\mathcal{D})$, then $([[es]]^{I}, [[es']]^{I})$ is also an arc in $G(\mathcal{G})$ since in GARBNs, any number of nodes can be synchronously updated. Thus, it is easy to imply that $[[FR^{\mathcal{D}}(\{es\})]]^{I} \subseteq FR^{\mathcal{G}}(\{[[es]]^{I}\})$. □

**Theorem 3.** *Let $\mathcal{D}$ be a DGARBN and $\mathcal{G}$ be its GARBN counterpart. Let $A^{\mathcal{D}}$ and $A^{\mathcal{G}}$ be the sets of attractors of $\mathcal{D}$ and $\mathcal{G}$, respectively. Then, there exists a mapping $m : A^{\mathcal{G}} \rightarrow A^{\mathcal{D}}$ with $[[m(att)]]^{I} \subseteq att$ for all $att \in A^{\mathcal{G}}$. Moreover, $m(att_1) \neq m(att_2)$ for all $att_1, att_2 \in A^{\mathcal{G}}, att_1 \neq att_2$. That means $m$ is an injection.*

*Proof.* Let $att \in A^{\mathcal{G}}$ and $s \in att$ be a state of $\mathcal{G}$. Obviously, $FR^{\mathcal{G}}(\{s\}) = att$.

Let $es'$ be an extended state of $\mathcal{D}$ such that $es'_i = s_i, i \in \{1, ..., n\}$ and $es'_{n+1} = 0$. By Lemma 1, we have $[[FR^{\mathcal{D}}(\{es'\})]]^{I} \subseteq FR^{\mathcal{G}}(\{[[es']]^{I}\}) = FR^{\mathcal{G}}(\{s\}) = att$. Clearly, there is an attractor $att'$ of $\mathcal{D}$ such that $att' \subseteq FR^{\mathcal{D}}(\{es'\})$. Then, $[[att']]^{I} \subseteq [[FR^{\mathcal{D}}(\{es'\})]]^{I}$ be the definition of image. Thus, $[[att']]^{I} \subseteq att$. We now can choose the mapping $m$ as $m(att) = att'$. Note that since $att'$ may not be unique, $m$ may not be uniquely determined.

For all $att_1, att_2 \in A^{\mathcal{G}}, att_1 \neq att_2$, we have $att_1 \cap att_2 = \emptyset$ since attractors of $\mathcal{G}$ are pairwise disjoint. $[[m(att_1)]]^{I} \subseteq att_1$ and $[[m(att_2)]]^{I} \subseteq att_2$ imply that $[[m(att_1)]]^{I} \cap [[m(att_2)]]^{I} = \emptyset$. Then, $m(att_1) \cap m(att_2) = \emptyset$ by the definition of image. Thus, $m(att_1) \neq m(att_2)$. Therefore, $m$ is an injection. □

**Theorem 4.** *Let $\mathcal{M}$ be an MxRBN of $M$ pure contexts. Let $\mathcal{D}$ be an arbitrary DGARBN among $M$ constituent DGARBNs of $\mathcal{M}$. Let $\mathcal{A}^{\mathcal{M}}$ and $\mathcal{A}^{\mathcal{D}}$ be the sets of attractors of $\mathcal{M}$ and $\mathcal{D}$, respectively. Then, there exists a mapping $m : \mathcal{A}^{\mathcal{M}} \rightarrow \mathcal{A}^{\mathcal{D}}$ with $m(att) \subseteq att$ for all $att \in \mathcal{A}^{\mathcal{M}}$. Moreover, $m(att_1) \neq m(att_2)$ for all $att_1, att_2 \in \mathcal{A}^{\mathcal{M}}, att_1 \neq att_2$. That means $m$ is an injection.*

*Proof.* Let $att \in \mathcal{A}^{\mathcal{M}}$. Obviously, an attractor of $\mathcal{M}$ always contains an extended state $es$ such that $es_{n+1} = 0$. Thus, there is an extended state $es$ where $es \in att$ and $es_{n+1} = 0$. Then, $FR^{\mathcal{M}}(\{es\}) = att$. Moreover, $es$ is also an extended state of $\mathcal{D}$.

Since the ESTG of $\mathcal{M}$ contains the ESTG of $\mathcal{D}$, $FR^{\mathcal{D}}(\{es\}) \subseteq FR^{\mathcal{M}}(\{es\})$. Obviously, there is an attractor $att'$ of $\mathcal{D}$ such that $att' \subseteq FR^{\mathcal{D}}(\{es\})$. Then, $att' \subseteq FR^{\mathcal{M}}(\{es\}) = att$. We now can choose the mapping $m$ as $m(att) = att'$. Note that since $att'$ may not be unique, $m$ may not be uniquely determined.

For all $att_1, att_2 \in \mathcal{A}^{\mathcal{M}}, att_1 \neq att_2$, we have $att_1 \cap att_2 = \emptyset$ since attractors of $\mathcal{M}$ are pairwise disjoint. $m(att_1) \subseteq att_1$ and $m(att_1) \subseteq att_1$ imply that $m(att_1) \cap m(att_2) = \emptyset$. Thus, $m(att_1) \neq m(att_2)$. Therefore, $m$ is an injection. □

**Theorem 5.** *The proposed SMT-based method terminates and correctly finds all attractors of a DGARBN.*

*Proof.* Let $\mathcal{M}$ and $\mathcal{D}$ denote the proposed SMT-based method and the DGARBN, respectively.

Until at least one attractor remains unmarked, we can find a path of any length such that its last extended state is not in any marked attractors since we can cycle in an attractor forever. That means $\mathcal{M}$ will not terminate if at least one attractor remains unmarked (a).

In the beginning of $\mathcal{M}$, the number of marked attractors is 0. The length of a path such that it does not contain any limit cycles must be less than the number of extended states of the ESTG of $\mathcal{D}$ (b). A better upper bound is the *diameter* of the ESTG. The diameter of a graph is defined as the length of the longest shortest path between two nodes. Since $p$ always increases when no limit cycle is found and the ESTG has at least one limit cycle, we eventually find a path containing a limit cycle by (a). Now this limit cycle is marked (i.e., is added to the set of marked attractors). If all attractors of $\mathcal{D}$ are marked, $\mathcal{M}$ must eventually terminate by (b). Otherwise, $\mathcal{M}$ continues its search by (a). Since the number of attractors of $\mathcal{D}$ is finite, all attractors will eventually be marked. Now, $\mathcal{M}$ can terminate by (b) and all attractors are found. $\square$